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Real forms and conjugacy classes of involutive automorphisms of the $D_\ell^{(1)}$ affine Kac–Moody algebras

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Abstract. The real forms of the complex affine Kac–Moody algebras $D_\ell^{(1)}$ (for $\ell \geqslant 4$) are found by determining all the conjugacy classes of the involutive automorphisms of $D_\ell^{(1)}$ using the matrix formulation of automorphisms of an affine Kac–Moody algebra.

1. Introduction

It is well known that the study of the involutive automorphisms of complex semi-simple Lie algebras by Gantmacher [1] allowed Gantmacher [2] to obtain a very elegant systematic determination of all the simple real Lie algebras. As the situation for the affine Kac–Moody algebras is similar, the determination the conjugacy classes of the group of automorphisms of these algebras will yield their real forms. An interesting alternative method, using Sakate diagrams, has been proposed recently by Pati and Parashar [3].

A comprehensive method of dealing with the automorphisms of an untwisted affine Kac–Moody algebra based on a matrix formulation of the untwisted affine Kac–Moody algebras was developed by Cornwell [4] (hereafter referred to as paper I), extending some previous work on the corresponding 'derived subalgebra' $\tilde{\mathcal{L}}'$ by Levstein [5]. It was shown in paper I that in general there are four types of automorphism within this matrix formulation, which were called 'type 1a', 'type 1b', 'type 2a' and 'type 2b'. The explicit derivation of all of these types of automorphism, as well as the investigation of identical automorphisms and of the identity automorphism, the formulae for the products of automorphisms, the conditions for an automorphism to be involutive, the formulae for the inverses of automorphisms and the conjugacy conditions for automorphisms may all be found in paper I, along with other motivations for the study of these automorphisms. These will not be repeated here. Previous papers (Cornwell [6–8], Clarke and Cornwell [9–11]) have discussed in detail the conjugacy classes of the involutive automorphisms of the *untwisted* algebras $A_\ell^{(1)}$ and $B_\ell^{(1)}$ (for $\ell \geqslant 1$) and for $C_\ell^{(1)}$ (for $\ell \geqslant 2$), and have also investigated the applicability of the method to the *twisted* algebras $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$). (It is interesting that the Sakate diagram method of Pati and Parashar [3] produces the results in complete agreement with these for the two cases of $A_1^{(1)}$ and $A_2^{(1)}$ that they have studied in detail.) In the present paper the investigation of the untwisted affine Kac–Moody algebras based on classical Lie algebras is completed by presenting for the remaining set of algebras $D_\ell^{(1)}$ (for $\ell \geqslant 4$) the conjugacy classes of the involutive automorphisms and their corresponding real forms.

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The complex affine Kac–Moody algebras and their Weyl groups have structures that are now very well established. For reviews see Kac [12], Goddard and Olive [13, 14] and Cornwell [15]. Unless otherwise stated all the notations and conventions that will be employed in the present paper are those of the latter reference. In particular, quantities belonging to the simple complex Lie algebra $\tilde{\mathcal{L}}^0$ associated with an untwisted affine Kac–Moody algebra $\tilde{\mathcal{L}}$ are distinguished from the corresponding quantities belonging to $\tilde{\mathcal{L}}$ by a superscript 0, so that, for example, α is the linear functional on the Cartan subalgebra $\mathcal{H}$ of $\tilde{\mathcal{L}}$ which is the extension of the linear functional α^0 on the Cartan subalgebra $\mathcal{H}^0$ of $\tilde{\mathcal{L}}^0$.

Mention should be made here of some other interesting papers on the automorphisms of complex affine Kac–Moody algebras and related real forms (although none deal *explicitly* with the special set of algebras $D_\ell^{(1)}$ (for $\ell \geqslant 4$) that are being considered here). The treatment of the so-called 'Cartan-preserving automorphisms', that is, of those automorphisms that map the Cartan subalgebra into itself, has been investigated by Bausch [16] and Gorman *et al* [17]. However, as discussed in paper I, although such automorphisms are very important, in that every conjugacy class of the automorphism group contains at least one Cartan-preserving automorphism, it is necessary to go beyond such automorphisms. The main reason for this is that it is possible for two Cartan-preserving automorphisms to be conjugate members of the group of all automorphisms of an affine Kac–Moody algebra, even though they are not conjugate within the subgroup of Cartan-preserving automorphisms. That is, conjugacy of Cartan-preserving automorphisms within the group of all automorphisms of an affine Kac–Moody algebra is often achieved via 'non-Cartan-preserving automorphisms'. The matrix method of paper I provides a systematic method of dealing with this problem. As mentioned above, this is a development of previous work by Levstein [5], which dealt with the corresponding problem for the derived subalgebra $\tilde{\mathcal{L}}'$ of an affine Kac–Moody algebra $\tilde{\mathcal{L}}$, that is, of the subalgebra of $\tilde{\mathcal{L}}$ with the scaling element d removed. Details of what is involved in this development can be found in paper I, but it is worth noting here that as the linear functional δ on the Cartan subalgebra $\mathcal{H}$ of $\tilde{\mathcal{L}}$ has *zero* value on the *whole* of the Cartan subalgebra of $\tilde{\mathcal{L}}'$, all the roots of $\tilde{\mathcal{L}}'$ have *infinite* degeneracy, whereas all the roots of $\tilde{\mathcal{L}}$ have *finite* degeneracy, and consequently $\tilde{\mathcal{L}}'$ and $\tilde{\mathcal{L}}$ have significantly different structures. Real forms of complex affine Kac–Moody algebras have also been discussed by Berman and Pianzola [18], Berman [19] and Rousseau [20].

The associated simple complex Lie algebra for $D_\ell^{(1)}$ is the algebra D_ℓ, so that

$$\delta = \alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell$$
$$c = h_{\alpha_0} + h_{\alpha_1} + 2(h_{\alpha_2} + \cdots + h_{\alpha_{\ell-2}}) + h_{\alpha_{\ell-1}} + h_{\alpha_\ell}.$$

For D_ℓ the basic representation $\mathbf{\Gamma}$ may be taken to be the 2ℓ-dimensional irreducible representation $\mathbf{\Gamma}\{1, 0, \ldots, 0\}$ which is defined as follows. Let

$$\alpha_j^\circ = \begin{cases} \varepsilon_j - \varepsilon_{j+1} & j = 1, 2, \ldots, \ell - 1 \\ \varepsilon_\ell + \varepsilon_{\ell+1} & j = \ell \end{cases}$$

so that the roots of D_ℓ may be expressed in the form $\varepsilon_r \pm \varepsilon_s$ for $1 \leqslant r < s \leqslant \ell$. Then the matrices representing the basis elements of D_ℓ may be taken to be

$$\mathbf{\Gamma}(h_{\alpha_j}) = \begin{cases} \{\ell(\ell-1)\}^{-1}\{\boldsymbol{e}_{k,k} - \boldsymbol{e}_{k+1,k+1} + \boldsymbol{e}_{2\ell-k,2\ell-k} - \boldsymbol{e}_{2\ell+1-k,2\ell+1-k}\} & \\ & \text{for} \quad j = 1, \ldots, \ell - 1 \\ \{\ell(\ell-1)\}^{-1}\{\boldsymbol{e}_{\ell-1,\ell-1} + \boldsymbol{e}_{\ell,\ell} - \boldsymbol{e}_{\ell+1,\ell+1} - \boldsymbol{e}_{\ell+2,\ell+2}\} & \text{for} \quad j = \ell \end{cases}$$

and

$$\mathbf{\Gamma}(e_{\varepsilon_r+\varepsilon_s}) = \{\ell(\ell-1)\}^{-1/2}\{\boldsymbol{e}_{r,2\ell+1-s} + (-1)^{r+s+1}\boldsymbol{e}_{s,2\ell+1-r}\} \qquad \text{for} \quad 1 \leqslant r < s \leqslant \ell$$
$$\mathbf{\Gamma}(e_{\varepsilon_r-\varepsilon_s}) = \{\ell(\ell-1)\}^{-1/2}\{\boldsymbol{e}_{r,s} + (-1)^{r+s+1}\boldsymbol{e}_{2\ell+1-s,2\ell+1-r}\} \qquad \text{for} \quad 1 \leqslant r < s \leqslant \ell$$

where the matrices $\boldsymbol{e}_{j,k}$ are defined in equation (A1) of the appendix. This representation $\boldsymbol{\Gamma}$ is equivalent to its contragredient representation, which implies that the type 1b automorphisms and the type 2b automorphisms coincide with the type 1a and the type 2a automorphisms, respectively. The Dynkin index of this representation is given by $\gamma = 2/\{\ell(\ell-1)\}$.

All the matrices $\boldsymbol{a}$ of this representation satisfy the constraint

$$\tilde{\boldsymbol{a}}\boldsymbol{g} + \boldsymbol{g}\boldsymbol{a} = \boldsymbol{0} \qquad \text{where} \quad \boldsymbol{g} = \text{offdiag}\{1, -1, \ldots, -1, 1\}. \tag{1}$$

If a matrix $\boldsymbol{U}(t)$ is to correspond to an automorphism of $D_\ell^{(1)}$, then the mapping $\boldsymbol{a}(t) \mapsto \boldsymbol{U}(t)\,\boldsymbol{a}(ut)\,\boldsymbol{U}(t)^{-1}$ must stabilize the 'matrix part' of the algebra. That is, the image of $\boldsymbol{a}(t)$ must satisfy (1). Schur's lemma and the Laurent polynomial condition then imply that $\tilde{\boldsymbol{U}}(t)\,\boldsymbol{g}\boldsymbol{U}(t) = \alpha t^\beta \boldsymbol{g}$, where α is some non-zero complex number and β is some integer.

Although the Dynkin diagram of D_4 is markedly different from that of D_ℓ for $\ell \geqslant 5$, this does not effect the *involutive* automorphisms.

A concise notation for certain matrices is presented in the appendix.

2. Type 1a involutive automorphisms with $u = 1$ and associated real forms of $D_\ell^{(1)}$

(1) The conjugacy class with representative given by $\boldsymbol{U}(t) = \boldsymbol{1}_{2\ell}$ consists only of the identity automorphism, for which the corresponding real form is just the compact real form.

(2) For each of the integer values of b such that $1 \leqslant b \leqslant [\ell/2]$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \text{dsum}\{\boldsymbol{1}_{\ell-b}, -\boldsymbol{1}_{2b}, \boldsymbol{1}_{\ell-b}\}$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

  (a) with eigenvalue +1:

   (i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$), $\mathrm{i}c$, $\mathrm{i}d$;

   (ii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

   (iii) $(e_\alpha + e_{-\alpha})$, $\mathrm{i}(e_\alpha - e_{-\alpha})$, together with $(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 2, \ldots$), where

$$\alpha = \varepsilon_r \pm \varepsilon_s \qquad \begin{cases} 1 \leqslant r < s \leqslant \ell - b \\ \ell - b < r < s \leqslant \ell \end{cases}$$

  (b) with eigenvalue -1:

   (i) $\mathrm{i}(e_\alpha + e_{-\alpha})$, $(e_\alpha - e_{-\alpha})$, together with $\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 2, \ldots$), where $\alpha = \varepsilon_r \pm \varepsilon_s$ (for $1 \leqslant r \leqslant \ell - b < s \leqslant \ell$).

(3) For each of the integer values of a such that $1 \leqslant a \leqslant [\ell/2]$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \text{dsum}\{-\boldsymbol{1}_a, \boldsymbol{1}_{\ell-a}, -\boldsymbol{1}_{\ell-a}, \boldsymbol{1}_a\}$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

  (a) with eigenvalue +1:

   (i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$), $\mathrm{i}c$, $\mathrm{i}d$;

   (ii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

   (iii) $(e_\alpha + e_{-\alpha})$, $\mathrm{i}(e_\alpha - e_{-\alpha})$, together with $(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 2, \ldots$), where $\alpha = \varepsilon_r + \varepsilon_s$ (for $1 \leqslant r < s < a$ or $a < r < s \leqslant \ell$);

   (iv) $(e_\alpha + e_{-\alpha})$, $\mathrm{i}(e_\alpha - e_{-\alpha})$, together with $(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 2, \ldots$), where $\alpha = \varepsilon_r - \varepsilon_s$ for $1 \leqslant r \leqslant a < s \leqslant \ell$);

(b) with eigenvalue -1:

(i) $\mathrm{i}(e_\alpha + e_{-\alpha})$, $(e_\alpha - e_{-\alpha})$, together with $\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 2, \ldots$), where $\alpha = \varepsilon_r + \varepsilon_s$ (for $1 \leqslant r \leqslant a < s \leqslant \ell$);

(ii) $\mathrm{i}(e_\alpha + e_{-\alpha})$, $(e_\alpha - e_{-\alpha})$, together with $\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 2, \ldots$), where $\alpha = \varepsilon_r + \varepsilon_s$ (for $1 \leqslant r < s < a$ or $a < r < s \leqslant \ell$).

(4) For ℓ *even* there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \mathrm{offsum}\{\boldsymbol{K}_{\ell-1}, t\boldsymbol{K}_1, t^{-1}\boldsymbol{K}_1, \boldsymbol{K}_{\ell-1}\}$, where $\boldsymbol{K}_j$ is the $j \times j$ matrix defined in (A2). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) h_{α_k} (for $k = 1, 2, \ldots, \ell - 2$), $(h_{\alpha_{\ell-1}} - \frac{1}{2}c)$, $(h_{\alpha_\ell} - \frac{1}{2}c)$, $\mathrm{i}c$, $\mathrm{i}\{d - \frac{1}{2}\ell(\ell-1)h_{\alpha_\ell}\}$;

(ii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

(iii) $\mathrm{i}(e_\alpha + e_{-\alpha})$ and $\mathrm{i}(e_\alpha - e_{-\alpha})$, together with

$$\left.\begin{array}{l} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array}\right. \qquad \text{(for } j = 1, 2, \ldots\text{)}$$

where $\alpha = \varepsilon_r \pm \varepsilon_s$ (for $1 \leqslant r < s \leqslant \ell - 1$, with $r + s$ odd);

(iv) $(e_\alpha + e_{-\alpha})$ and $\mathrm{i}(e_\alpha - e_{-\alpha})$, together with

$$\left.\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array}\right. \qquad \text{(for } j = 1, 2, \ldots\text{)}$$

where $\alpha = \varepsilon_r \pm \varepsilon_s$ (for $1 \leqslant r < s \leqslant \ell - 1$, with $r + s$ even);

(v) with $\alpha = \varepsilon_r \pm \varepsilon_\ell$ (for $1 \leqslant r \leqslant \ell - 1$, with r odd),

$$\left.\begin{array}{l} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \end{array}\right. \qquad \text{(for } j = 0, 1, 2, \ldots\text{)}$$

(vi) with $\alpha = \varepsilon_r \pm \varepsilon_\ell$ (for $1 \leqslant r \leqslant \ell - 1$, with r even),

$$\left.\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}). \end{array}\right. \qquad \text{(for } j = 0, 1, 2, \ldots\text{)}$$

(5) For ℓ *odd* there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \boldsymbol{K}_{2\ell}$, where $\boldsymbol{K}_j$ is the $j \times j$ matrix defined in (A2). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(a) with eigenvalue +1:

(i) $\mathrm{i}c$, $\mathrm{i}d$;

(ii) with

$$\alpha = \varepsilon_r \pm \varepsilon_s \qquad 1 \leqslant r < s \leqslant \ell \qquad \text{(with } r+s \text{ odd)} \tag{2}$$

$$\left.\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 1, 2, \ldots)$$

(iii) with

$$\alpha = \varepsilon_r \pm \varepsilon_s \qquad 1 \leqslant r < s \leqslant \ell \qquad \text{(with } r+s \text{ even)} \tag{3}$$

$$\left.\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 1, 2, \ldots)$$

(b) with eigenvalue -1:

(i) h_{α_k} (for $k = 1, 2, \ldots, \ell$);

(ii) $\mathrm{i}(e^k_{j\delta} + e^k_{-j\delta})$ and $(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

(iii) with α defined in (2),

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 1, 2, \ldots)$$

(iv) with α defined in (3),

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}).\end{array}\right. \qquad (\text{for } j = 1, 2, \ldots)$$

(6) For each of the integer values of a such that $1 \leqslant a \leqslant [\ell/2]$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \mathrm{dsum}\{t\boldsymbol{K}_{\ell-a}, \boldsymbol{K}_a, t\boldsymbol{K}_a, \boldsymbol{K}_{\ell-a}\}$, where $\boldsymbol{K}_j$ is the $j \times j$ matrix defined in (A2). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(a) with eigenvalue +1:

(i) $\mathrm{i}c$;

(ii) for $a > 2$,

$$\mathrm{i}d - \tfrac{1}{4}\mathrm{i}\left\{\sum_{p=1}^{\ell-a} p h_{\alpha_p} + \sum_{p=1}^{a-2}(\ell - a - p) h_{\alpha_{\ell-a-p}} + \tfrac{1}{2}(\ell - 2a + 2) h_{\alpha_{\ell-1}} + \tfrac{1}{2}(\ell - 2a) h_{\alpha_\ell}\right\}$$

for $a = 2$,

$$\mathrm{i}d - \tfrac{1}{4}\mathrm{i}\left\{\sum_{p=1}^{\ell-a} p h_{\alpha_p} + \tfrac{1}{2}(\ell - 2) h_{\alpha_{\ell-1}} + \tfrac{1}{2}(\ell - 4) h_{\alpha_\ell}\right\}$$

for $a = 1$,

$$\mathrm{i}d - \tfrac{1}{4}\mathrm{i}\left\{\sum_{p=1}^{\ell-a} p h_{\alpha_p} + \tfrac{1}{2}\ell h_{\alpha_{\ell-1}} + \tfrac{1}{2}(\ell - 2) h_{\alpha_\ell}\right\}$$

for $a = 0$,

$$\mathrm{i}d - \tfrac{1}{4}\mathrm{i}\left\{\sum_{p=1}^{\ell-a} p h_{\alpha_p} + \tfrac{1}{2}(\ell - 2) h_{\alpha_{\ell-1}} + \tfrac{1}{2}\ell h_{\alpha_\ell}\right\}$$

(iii) with

$$\alpha = \varepsilon_r - \varepsilon_s \qquad 1 \leqslant r < s \leqslant \ell - a \quad \text{or} \quad \ell - a < r < s \leqslant \ell \quad (\text{with } r+s \text{ even}) \tag{4}$$

or

$$\alpha = \varepsilon_r + \varepsilon_s \qquad 1 \leqslant r \leqslant \ell - a < s \leqslant \ell \qquad (\text{with } r+s \text{ even}) \tag{5}$$

$$\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})\end{array} \qquad (\text{for } j = 1, 2, \ldots)$$

(iv) with

$$\alpha = \varepsilon_r - \varepsilon_s \qquad 1 \leqslant r < s \leqslant \ell - a \quad \text{or} \quad \ell - a < r < s \leqslant \ell \quad (\text{with } r+s \text{ odd}) \tag{6}$$

or

$$\alpha = \varepsilon_r + \varepsilon_s \qquad 1 \leqslant r \leqslant \ell - a < s \leqslant \ell \qquad (\text{with } r+s \text{ odd}) \tag{7}$$

$$\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha})\end{array} \qquad (\text{for } j = 1, 2, \ldots)$$

(v) with

$$\alpha = \varepsilon_r - \varepsilon_s \qquad 1 \leqslant r \leqslant \ell - a < s \leqslant \ell \qquad (\text{with } r+s \text{ even}) \tag{8}$$

or

$$\alpha = \varepsilon_r + \varepsilon_s \qquad 1 \leqslant r < s \leqslant \ell - a \qquad (\text{with } r+s \text{ even}) \tag{9}$$

$$\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})\end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(vi) with

$$\alpha = \varepsilon_r - \varepsilon_s \qquad 1 \leqslant r \leqslant \ell - a < s \leqslant \ell \qquad (\text{with } r+s \text{ odd}) \tag{10}$$

or

$$\alpha = \varepsilon_r + \varepsilon_s \qquad 1 \leqslant r < s \leqslant \ell - a \qquad (\text{with } r+s \text{ odd}) \tag{11}$$

$$\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(vii) with

$$\alpha = \varepsilon_r + \varepsilon_s \qquad \ell - a < r < s \leqslant \ell \qquad (\text{with } r+s \text{ even}) \tag{12}$$

$$\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha})\end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(viii) with

$$\alpha = \varepsilon_r + \varepsilon_s \qquad \ell - a < r < s \leqslant \ell \qquad (\text{with } r+s \text{ odd}) \tag{13}$$

$$\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha})\end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(b) with eigenvalue -1:

(i) h_{α_k} (for $k = 1, 2, \ldots, \ell, k \neq \ell - a$), $(h_{\alpha_{\ell-a}} + \frac{1}{2}c)$;

(ii) $\mathrm{i}(e^k_{j\delta} + e^k_{-j\delta})$ and $(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

(iii) with α defined in (4) or (5),

$$\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha})\end{array} \qquad (\text{for } j = 1, 2, \ldots)$$

(iv) with α defined in (6) or (7),

$$\begin{array}{ll} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) & \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) & \text{(for } j = 1, 2, \ldots) \end{array}$$

(v) with α defined in (8) or (9),

$$\begin{array}{ll} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) & \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) & \text{(for } j = 0, 1, 2, \ldots) \end{array}$$

(vi) with α defined in (10) or (11),

$$\begin{array}{ll} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) & \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) & \text{(for } j = 0, 1, 2, \ldots) \end{array}$$

(vii) with α defined in (12),

$$\begin{array}{ll} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) & \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) & \text{(for } j = 0, 1, 2, \ldots) \end{array}$$

(viii) with α defined in (13),

$$\begin{array}{ll} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) & \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}). & \text{(for } j = 0, 1, 2, \ldots) \end{array}$$

(7) For integer values of b such that $1 \leqslant b \leqslant [\ell/2]$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \mathrm{dsum}\{\mathbf{1}_{\ell-b}, -\mathbf{1}_b, \boldsymbol{K}_2, -\mathbf{1}_b, -\mathbf{1}_{\ell-b}\}$, where $\boldsymbol{K}_j$ is the $j \times j$ matrix defined in (A2). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(a) with eigenvalue +1:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell - 2$), $\mathrm{i}(h_{\alpha_{\ell-1}} + h_{\alpha_\ell})$, $\mathrm{i}c$, $\mathrm{i}d$;

(ii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell - 2$ and $j = 1, 2, \ldots$);

(iii) $(e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} + e^\ell_{j\delta} + e^\ell_{-j\delta})$ and $\mathrm{i}(e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} + e^\ell_{j\delta} - e^\ell_{-j\delta})$ (for $j = 1, 2, \ldots$);

(iv) $(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$, where $\alpha = \varepsilon_r \pm \varepsilon_s$ (for $1 \leqslant r < s \leqslant \ell-1-b$ or $\ell - b - 1 < r < s \leqslant \ell - 1$);

(v) $(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta})$ (for $j = 0, 1, 2, \ldots$), where

$$\alpha = \varepsilon_r - \varepsilon_\ell \qquad \beta = \varepsilon_r + \varepsilon_\ell \qquad \text{with} \quad 1 \leqslant r \leqslant \ell - 1 - b \tag{14}$$

(vi) $(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta})$ (for $j = 0, 1, 2, \ldots$), where

$$\alpha = \varepsilon_r - \varepsilon_\ell, \beta = \varepsilon_r + \varepsilon_\ell \qquad \text{with} \quad \ell - 1 - b \leqslant r \leqslant \ell - 1 \tag{15}$$

(b) with eigenvalue −1:

(i) $(h_{\alpha_{\ell-1}} - h_{\alpha_\ell})$;

(ii) $\mathrm{i}(e^k_{j\delta} + e^k_{-j\delta})$ and $(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

(iii) $(e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} - e^\ell_{j\delta} - e^\ell_{-j\delta})$ and $\mathrm{i}(e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} - e^\ell_{j\delta} + e^\ell_{-j\delta})$ (for $j = 1, 2, \ldots$);

(iv) $(e_{j\delta+\alpha}+e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha})$, where $\alpha=\varepsilon_r\pm\varepsilon_s$ (for $1\leqslant r\leqslant \ell-1-b<s\leqslant\ell-1$);

(v) $\mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{j\delta+\beta}-e_{-j\delta-\beta})$ and $(e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{j\delta+\beta}+e_{-j\delta-\beta})$ (for $j=0,1,2,\ldots$), where α and β are given by (14);

(vi) $(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{j\delta+\beta}+e_{-j\delta-\beta})$ and $\mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{j\delta+\beta}-e_{-j\delta-\beta})$ (for $j=0,1,2,\ldots$), where α and β are given by (15).

(8) For each of the integer values of b that are such that $1\leqslant b\leqslant[\ell/2]$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t)=\mathrm{dsum}\{\mathbf{1}_{\ell-2-b},-\mathbf{1}_b,\mathrm{offdiag}\{1,t,t^{-1},1\},-\mathbf{1}_b,\mathbf{1}_{\ell-2-b}\}$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k=1,2,\ldots,\ell-2$), $(h_{\alpha_{\ell-1}}-\frac{1}{2}c)$, $(h_{\alpha_\ell}+\frac{1}{2}c)$, $\mathrm{i}c$, $\mathrm{i}\{d+\frac{1}{4}\ell(\ell-1)(h_{\alpha_{\ell-1}}-h_{\alpha_\ell})\}$;
(ii) $(e^k_{j\delta}+e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta}-e^k_{-j\delta})$ (for $k=1,2,\ldots,\ell-2$, and $j=1,2,\ldots$);
(iii) $\mathrm{i}(e^k_{j\delta}+e^k_{-j\delta})$ and $(e^k_{j\delta}-e^k_{-j\delta})$ (for $k=\ell-1,\ell$ and $j=1,2,\ldots$);
(iv) for $j=1,2,\ldots,$

$$\{e^{\ell-2}_{j\delta}+e^{\ell-2}_{-j\delta}+\tfrac{1}{2}(e^{\ell-1}_{j\delta}+e^{\ell-1}_{-j\delta}+e^{\ell}_{j\delta}+e^{\ell}_{-j\delta})\}$$
$$\{e^{\ell-2}_{j\delta}-e^{\ell-2}_{-j\delta}-\tfrac{1}{2}(e^{\ell-1}_{j\delta}-e^{\ell-1}_{-j\delta}+e^{\ell}_{j\delta}-e^{\ell}_{-j\delta})\}$$

(v) $(e_{j\delta+\alpha}+e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha})$ (for $j=1,2,\ldots$), where $\alpha=\varepsilon_r\pm\varepsilon_s$ (for $1\leqslant r<s\leqslant\ell-2-b$ or $\ell-2-b<r<s\leqslant\ell-2$);

(vi) $\mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha})$ and $(e_{j\delta+\alpha}-e_{-j\delta-\alpha})$ (for $j=1,2,\ldots$), where $\alpha=\varepsilon_r\pm\varepsilon_s$ (for $1\leqslant r\leqslant\ell-2-b<s\leqslant\ell-2$);

(vii) with $\alpha=\varepsilon_r-\varepsilon_{\ell-1}$ and $\beta=\varepsilon_r+\varepsilon_{\ell-1}$ (with $1\leqslant r\leqslant\ell-2-b$),

$$\left.\begin{array}{l}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{j\delta+\beta}+e_{-j\delta-\beta})\\ \mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{j\delta+\beta}-e_{-j\delta-\beta})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{j\delta+\beta}-e_{-j\delta-\beta})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{j\delta+\beta}+e_{-j\delta-\beta})\end{array}\right.\qquad (j=0,1,2,\ldots)$$

(viii) with $\alpha=\varepsilon_r-\varepsilon_{\ell-1}$ and $\beta=\varepsilon_r+\varepsilon_{\ell-1}$ (with $\ell-2-b<r\leqslant\ell-2$),

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{j\delta+\beta}+e_{-j\delta-\beta})\\ (e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{j\delta+\beta}-e_{-j\delta-\beta})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{j\delta+\beta}-e_{-j\delta-\beta})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{j\delta+\beta}+e_{-j\delta-\beta})\end{array}\right.\qquad (j=0,1,2,\ldots)$$

(ix) with $\alpha=\varepsilon_r-\varepsilon_\ell$ and $\beta=\varepsilon_r+\varepsilon_\ell$ (with $1\leqslant r\leqslant\ell-2-b$),

$$\left.\begin{array}{l}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{(j+1)\delta+\beta}+e_{-(j+1)\delta-\beta})\\ \mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{(j+1)\delta+\beta}-e_{-(j+1)\delta-\beta})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{(j+1)\delta+\beta}-e_{-(j+1)\delta-\beta})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{(j+1)\delta+\beta}+e_{-(j+1)\delta-\beta})\end{array}\right.\qquad (j=0,1,2,\ldots)$$

(x) with $\alpha=\varepsilon_r-\varepsilon_\ell$ and $\beta=\varepsilon_r+\varepsilon_\ell$ (with $\ell-2-b<r\leqslant\ell-2$),

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{(j+1)\delta+\beta}+e_{-(j+1)\delta-\beta})\\ (e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{(j+1)\delta+\beta}-e_{-(j+1)\delta-\beta})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{(j+1)\delta+\beta}-e_{-(j+1)\delta-\beta})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{(j+1)\delta+\beta}+e_{-(j+1)\delta-\beta})\end{array}\right.\qquad (j=0,1,2,\ldots)$$

(xi) with $\alpha = \varepsilon_{\ell-1} \pm \varepsilon_\ell$,

$$\begin{array}{ll} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\alpha} + e_{-(j+1)\delta+\alpha}) & \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\alpha} + e_{-(j+1)\delta+\alpha}) & \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\alpha} + e_{-(j+1)\delta+\alpha}) & (j = 0, 1, 2, \ldots) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\alpha} + e_{-(j+1)\delta+\alpha}). & \end{array}$$

3. Type 1a involutive automorphisms with $u = -1$ and associated real forms of $D_\ell^{(1)}$

(1) There exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \mathbf{1}_{2\ell}$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(a) with eigenvalue +1:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$), ic, id;

(ii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 2, 4, \ldots$, i.e. for j even);

(iii) $(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 2, 4, \ldots$, i.e. for j even), where α is any root of D_ℓ;

(b) with eigenvalue -1:

(i) $\mathrm{i}(e^k_{j\delta} + e^k_{-j\delta})$ and $(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 3, \ldots$, i.e. for j odd);

(ii) $\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 3, \ldots$, i.e. for j odd), where α is any root of D_ℓ.

(2) For l *odd* there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \mathrm{dsum}\{\mathbf{1}_{\ell-1}, \boldsymbol{K}_2, \mathbf{1}_{\ell-1}\}$, where $\boldsymbol{K}_j$ is the $j \times j$ matrix defined in (A2). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$), ic, id;

(ii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell - 1$, and $j = 2, 4, \ldots$, i.e. for j even);

(iii) $\mathrm{i}(e^k_{j\delta} + e^k_{-j\delta})$ and $(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell - 1$, and $j = 1, 3, \ldots$, i.e. for j odd);

(iv) for $j = 2, 4, \ldots$, i.e. for j even,

$$\begin{array}{l} (e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} + e^{\ell}_{j\delta} + e^{\ell}_{-j\delta}) \\ \mathrm{i}(e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} - e^{\ell}_{j\delta} - e^{\ell}_{-j\delta}) \\ \mathrm{i}(e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} + e^{\ell}_{j\delta} - e^{\ell}_{-j\delta}) \\ (e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} - e^{\ell}_{j\delta} + e^{\ell}_{-j\delta}) \end{array}$$

(v) for $j = 1, 3, \ldots$, i.e. for j odd,

$$\begin{array}{l} \mathrm{i}(e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} + e^{\ell}_{j\delta} + e^{\ell}_{-j\delta}) \\ (e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} - e^{\ell}_{j\delta} - e^{\ell}_{-j\delta}) \\ (e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} + e^{\ell}_{j\delta} - e^{\ell}_{-j\delta}) \\ \mathrm{i}(e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} - e^{\ell}_{j\delta} + e^{\ell}_{-j\delta}) \end{array}$$

(vi) $(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 2, 4, \ldots$, i.e. for j even), where $\alpha = \varepsilon_r \pm \varepsilon_s$ (for $1 \leqslant r < s \leqslant \ell - 1$);

(vii) $\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 3, \ldots$, i.e. for j odd), where $\alpha = \varepsilon_r \pm \varepsilon_s$ (for $1 \leqslant r < s \leqslant \ell - 1$);

(viii) $\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha})$ and $(e_{j\delta+\alpha} - e_{-j\delta-\alpha})$ (for $j = 1, 2, \ldots$), where $\alpha = \varepsilon_r \pm \varepsilon_s$ (for $1 \leqslant r \leqslant \ell - 2 - b < s \leqslant \ell - 2$);

(ix) with $\alpha = \varepsilon_r - \varepsilon_\ell$ and $\beta = \varepsilon_r + \varepsilon_\ell$ (with $1 \leqslant r \leqslant \ell - 1$),

$$\left.\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}) \end{array}\right\} \quad (j \text{ even})$$

(x) with $\alpha = \varepsilon_r - \varepsilon_\ell$ and $\beta = \varepsilon_r + \varepsilon_\ell$ (with $1 \leqslant r \leqslant \ell - 1$),

$$\left.\begin{array}{l} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}). \end{array}\right\} \quad (j \text{ odd})$$

4. Type 2a involutive automorphisms with $u = 1$ and associated real forms of $D^{(1)}_\ell$

(1) For each pair of the integer values of q and r that are such that $0 \leqslant q \leqslant [\ell/2]$ and $0 \leqslant r \leqslant \left[\frac{1}{2}(\ell - q - 1)\right]$ there exists a class of involutive automorphisms with a representative given by $\boldsymbol{U}(t) = \mathrm{dsum}\{\mathbf{1}_p, -\mathbf{1}_q, t\mathbf{1}_r, t^{-1}\mathbf{1}_r, -\mathbf{1}_q.\mathbf{1}_p\}$, where $p + q + r = \ell$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$ and if $r \neq 0$ then $k \neq \ell - r$), $\mathrm{i}(h_{\alpha_{\ell-r}} - \frac{1}{2}c)$ (for $r \neq 0$), $\mathrm{i}(h_{\alpha_\ell} + \frac{1}{2}c)$ (for $r = 1$), $\mathrm{i}(h_{\alpha_\ell} + c)$ (for $r = 2$);

(ii) $c, d - \frac{1}{2}\ell(\ell-1)\left\{\sum_{j=1, r\geqslant 2}^{r-2} jh_{\alpha_{\ell-r+r}} - \frac{1}{2}(r-2)h_{\alpha_{\ell-1}} - \frac{1}{2}(r-4)h_{\alpha_\ell}\right\}$;

(iii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

(iv) with $\alpha = \varepsilon_a \pm \varepsilon_b$ $1 \leqslant a < b \leqslant p$ or $p < a < b \leqslant p + q$, or with $\alpha = \varepsilon_a - \varepsilon_b$, $p + q < a < b \leqslant \ell$,

$$\left.\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array}\right\} \quad (\text{for } j = 1, 2, \ldots)$$

(v) with $\alpha = \varepsilon_a \pm \varepsilon_b$, $1 \leqslant a \leqslant p < b \leqslant p+q$,

$$\left.\begin{array}{l} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array}\right. \qquad (\text{for } j = 1, 2, \ldots)$$

(vi) with $\alpha = \varepsilon_a - \varepsilon_b$, $1 \leqslant a \leqslant p$, $p+q < b \leqslant \ell$,

$$\left.\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(vii) with $\alpha = \varepsilon_a - \varepsilon_b$, $p < a \leqslant p+q < b \leqslant \ell$,

$$\left.\begin{array}{l} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(viii) with $\alpha = \varepsilon_a + \varepsilon_b$, $p < a \leqslant p$, $p+q < b \leqslant \ell$,

$$\left.\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(ix) with $\alpha = \varepsilon_a + \varepsilon_b$, $p < a \leqslant p+q < b \leqslant \ell$,

$$\left.\begin{array}{l} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(x) with $\alpha = \varepsilon_a + \varepsilon_b$, $p+q < a < b \leqslant \ell$,

$$\left.\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}). \end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(2) For each pair of the non-negative integer values of p, q and r that are such that $p + q + r = \ell$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \mathrm{dsum}\{\mathbf{1}_p, \mathbf{1}_q, t\mathbf{1}_r, -t^{-1}\mathbf{1}_r, -\mathbf{1}_q, -\mathbf{1}_p\}$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$ and if $r \neq 0$ then $k \neq \ell - r$), $\mathrm{i}(h_{\alpha_{\ell-r}} - \frac{1}{2}c)$ (for $r \neq 0$), $\mathrm{i}(h_{\alpha_\ell} + \frac{1}{2}c)$ (for $r = 1$), $\mathrm{i}(h_{\alpha_\ell} + c)$ (for $r = 2$);

(ii) c, $d - \frac{1}{2}\ell(\ell-1)\left\{\sum_{j=1,r\geqslant 2}^{r-2} jh_{\alpha_{\ell-r+j}} - \frac{1}{2}(r-2)h_{\alpha_{\ell-1}} - \frac{1}{2}(r-4)h_{\alpha_\ell}\right\}$;

(iii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots$);

(iv) with $\alpha = \varepsilon_a - \varepsilon_b$, $1 \leqslant a < b \leqslant p$, $p < a < b \leqslant p + q$, or $p + q < a < b \leqslant \ell$, or with $\alpha = \varepsilon_a + \varepsilon_b$, $1 \leqslant a \leqslant p < b \leqslant p + q$,

$$\left.\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 1, 2, \ldots)$$

(v) with $\alpha = \varepsilon_a - \varepsilon_b$, $1 \leqslant a \leqslant p < b \leqslant p + q$, or with $\alpha = \varepsilon_a + \varepsilon_b$, $1 \leqslant a < b \leqslant p$, or $p < a < b \leqslant p + q$,

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha})\\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 1, 2, \ldots)$$

(vi) with $\alpha = \varepsilon_a - \varepsilon_b$, $1 \leqslant a \leqslant p$, $p + q < b \leqslant \ell$,

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha})\\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha})\\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(vii) with $\alpha = \varepsilon_a - \varepsilon_b$, $p < a \leqslant p + q < b \leqslant \ell$,

$$\left.\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha})\\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(viii) with $\alpha = \varepsilon_a + \varepsilon_b$, $1 \leqslant a \leqslant p$, $p + q < b \leqslant \ell$,

$$\left.\begin{array}{l}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(ix) with $\alpha = \varepsilon_a + \varepsilon_b$, $p < a \leqslant p + q < b \leqslant \ell$

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})\\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})\end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(x) with $\alpha = \varepsilon_a + \varepsilon_b$, $p + q < a < b \leqslant \ell$,

$$\left.\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha})\\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha})\\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}).\end{array}\right. \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(3) For each pair of the non-negative integer values of p, q and r that are such that $p+q+r=\ell$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t)=\mathrm{dsum}\{t\mathbf{1}_p, -\mathbf{1}_q, \mathbf{1}_r, t\mathbf{1}_r, -t\mathbf{1}_q, \mathbf{1}_p\}$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k=1,2,\ldots,\ell$, with $k\neq p$), $\mathrm{i}(h_{\alpha_p}+\frac{1}{2}c)$, c;

(ii) for $p\neq\ell-1$ or ℓ:

$$d-\tfrac{1}{4}\ell(\ell-1)\left\{\sum_{j=1}^{p} jh_{\alpha_j}+\sum_{j=1}^{\ell-2-p} h_{\alpha_{p+j}}+\tfrac{1}{2}(2p+2-\ell)(h_{\alpha_{\ell-1}}+h_{\alpha_\ell})\right\}$$

for $p=\ell-1$:

$$d-\tfrac{1}{4}\ell(\ell-1)\left\{\sum_{j=1}^{\ell-2} jh_{\alpha_j}+\tfrac{1}{2}\ell h_{\alpha_{\ell-1}}+\tfrac{1}{2}(\ell-2)h_{\alpha_\ell}\right\}$$

and for $p=\ell$:

$$d-\tfrac{1}{4}\ell(\ell-1)\left\{\sum_{j=1}^{\ell-2} jh_{\alpha_j}+\tfrac{1}{2}(\ell-2)h_{\alpha_{\ell-1}}+\tfrac{1}{2}\ell h_{\alpha_\ell}\right\}$$

(iii) $(e^k_{j\delta}+e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta}-e^k_{-j\delta})$ (for $k=1,2,\ldots,\ell$ and $j=1,2,\ldots$);

(iv) with $\alpha=\varepsilon_a-\varepsilon_b,\ 1\leqslant a<b\leqslant p$, $p<a<b\leqslant p+q$, or $p+q<a<b\leqslant\ell$, or with $\alpha=\varepsilon_a+\varepsilon_b,\ 1\leqslant a\leqslant p$, $p+q<b\leqslant\ell$,

$$\begin{array}{l}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{-j\delta+\alpha}+e_{j\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{-j\delta+\alpha}-e_{j\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{-j\delta+\alpha}-e_{j\delta-\alpha})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{-j\delta+\alpha}+e_{j\delta-\alpha})\end{array}\qquad (\text{for } j=1,2,\ldots)$$

(v) with $\alpha=\varepsilon_a-\varepsilon_b$, $p<a\leqslant p+q<b\leqslant\ell$, or with $\alpha=\varepsilon_a+\varepsilon_b,\ 1\leqslant a\leqslant p<b\leqslant p+q$,

$$\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{-j\delta+\alpha}+e_{j\delta-\alpha})\\ (e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{-j\delta+\alpha}-e_{j\delta-\alpha})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{-j\delta+\alpha}-e_{j\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{-j\delta+\alpha}+e_{j\delta-\alpha})\end{array}\qquad (\text{for } j=1,2,\ldots)$$

(vi) with $\alpha=\varepsilon_a+\varepsilon_b$, $p<a<b\leqslant p+q$ or $p+q<a<b\leqslant\ell$,

$$\begin{array}{l}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{-(j+1)\delta+\alpha}+e_{(j+1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{-(j+1)\delta+\alpha}-e_{(j+1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{-(j+1)\delta+\alpha}-e_{(j+1)\delta-\alpha})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{-(j+1)\delta+\alpha}+e_{(j+1)\delta-\alpha})\end{array}\qquad (\text{for } j=0,1,2,\ldots)$$

(vii) with $\alpha=\varepsilon_a+\varepsilon_b$, $p<a\leqslant p+q<b\leqslant\ell$,

$$\begin{array}{l}\mathrm{i}(e_{j\delta+\alpha}+e_{-j\delta-\alpha}+e_{-(j+1)\delta+\alpha}+e_{(j+1)\delta-\alpha})\\ (e_{j\delta+\alpha}+e_{-j\delta-\alpha}-e_{-(j+1)\delta+\alpha}-e_{(j+1)\delta-\alpha})\\ (e_{j\delta+\alpha}-e_{-j\delta-\alpha}+e_{-(j+1)\delta+\alpha}-e_{(j+1)\delta-\alpha})\\ \mathrm{i}(e_{j\delta+\alpha}-e_{-j\delta-\alpha}-e_{-(j+1)\delta+\alpha}+e_{(j+1)\delta-\alpha})\end{array}\qquad (\text{for } j=0,1,2,\ldots)$$

(viii) with $\alpha = \varepsilon_a - \varepsilon_b$, $1 \leqslant a \leqslant p < b \leqslant p+q$,

$$\begin{aligned}
&\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})\\
&(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\
&(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\
&\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})
\end{aligned} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(ix) with $\alpha = \varepsilon_a - \varepsilon_b$, $1 \leqslant a \leqslant p$, $p+q < b \leqslant \ell$ and $1 \leqslant a < b \leqslant p$,

$$\begin{aligned}
&(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha})\\
&\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\
&\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha})\\
&(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}).
\end{aligned} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(4) For each pair of the non-negative integer values of p, q and r that are such that $p+q+r = \ell-1$ there exists a class of involutive automorphisms with representative given by $\boldsymbol{U}(t) = \mathrm{dsum}\{\mathbf{1}_p, -\mathbf{1}_q, t\mathbf{1}_r, \boldsymbol{K}_2, t^{-1}\mathbf{1}_r, -\mathbf{1}_q, \mathbf{1}_p\}$, where $\boldsymbol{K}_j$ is the $j \times j$ matrix defined in (A2). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) $\mathrm{i}h_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell-2$, with $k \neq \ell-r-2$), $\mathrm{i}(h_{\alpha_{\ell-r-2}} - \frac{1}{2}c)$ (for $r \neq 0$), $\mathrm{i}(h_{\alpha_{\ell-1}} + h_{\alpha_\ell} + c)$ (for $r \neq 0$), $\mathrm{i}(h_{\alpha_{\ell-1}} + h_{\alpha_\ell})$ (for $r = 0$), $(h_{\alpha_{\ell-1}} - h_{\alpha_\ell})$, c;

(ii) $d - \frac{1}{4}\left\{\sum_{j=1}^{r-1} j h_{\alpha_{\ell-r+1+j}} + \frac{1}{2}(r+1)(h_{\alpha_{\ell-1}} + h_{\alpha_\ell})\right\}$ (for $r > 0$), d (for $r = 0$);

(iii) $(e^k_{j\delta} + e^k_{-j\delta})$ and $\mathrm{i}(e^k_{j\delta} - e^k_{-j\delta})$ (for $k = 1, 2, \ldots, \ell-2$, and $j = 1, 2, \ldots$);

(iv) with $j = 1, 2, \ldots$

$$\begin{aligned}
&(e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} + e^{\ell}_{j\delta} + e^{\ell}_{-j\delta})\\
&(e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} + e^{\ell}_{j\delta} - e^{\ell}_{-j\delta})\\
&\mathrm{i}(e^{\ell-1}_{j\delta} + e^{\ell-1}_{-j\delta} - e^{\ell}_{j\delta} - e^{\ell}_{-j\delta})\\
&\mathrm{i}(e^{\ell-1}_{j\delta} - e^{\ell-1}_{-j\delta} - e^{\ell}_{j\delta} + e^{\ell}_{-j\delta})
\end{aligned}$$

(v) with $\alpha = \varepsilon_a \pm \varepsilon_b$, $1 \leqslant a < b \leqslant p$, or $p < a < b \leqslant p+q$, or with $\alpha = \varepsilon_a - \varepsilon_b$, $p+q < a < b \leqslant \ell-2$,

$$\begin{aligned}
&(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha})\\
&\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\
&\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\
&(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})
\end{aligned} \qquad (\text{for } j = 1, 2, \ldots)$$

(vi) with $\alpha = \varepsilon_a \pm \varepsilon_b$, $1 \leqslant a \leqslant p < b \leqslant p+q$,

$$\begin{aligned}
&\mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha})\\
&(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\
&(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha})\\
&\mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})
\end{aligned} \qquad (\text{for } j = 1, 2, \ldots)$$

(vii) with $\alpha = \varepsilon_a - \varepsilon_b$, $1 \leqslant a \leqslant p$, $p+q < b \leqslant \ell-2$,

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(viii) with $\alpha = \varepsilon_a - \varepsilon_b$, $p < a \leqslant p+q < b \leqslant \ell-2$,

$$\begin{array}{l} \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}) \end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(ix) with $\alpha = \varepsilon_a + \varepsilon_b$, $1 \leqslant a \leqslant p$, $p+q < b \leqslant \ell-2$,

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(x) with $\alpha = \varepsilon_a + \varepsilon_b$, $p < a \leqslant p+q < b \leqslant \ell-2$,

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}) \end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

(xi) with $\alpha = \varepsilon_a + \varepsilon_b$, $p+q < a < b \leqslant \ell-2$,

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}) \\ \mathrm{i}(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}). \end{array} \qquad (\text{for } j = 0, 1, 2, \ldots)$$

Appendix. Concise notation for matrices

Some notation is defined here to make the preceding analysis more concise. The expressions ‘dsum’ and ‘offsum’ are analogous to the commonly used ‘diag’.

(1) The first of these indicates a direct sum. For example

$$\mathrm{dsum}\{\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{y}, \boldsymbol{z}\} = \begin{pmatrix} \boldsymbol{a} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{z} \end{pmatrix}$$

where $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{y}, \boldsymbol{z}$ are square matrices.

(2) The expression 'offsum' is similar to the previous expression. Thus

$$\mathrm{offsum}\{\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{y}, \boldsymbol{z}\} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{b} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \boldsymbol{y} & \cdots & \mathbf{0} & \mathbf{0} \\ \boldsymbol{z} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{y}, \boldsymbol{z}$ are all square matrices.

(3) The expression 'offdiag' is a special case of the previous expression. Thus

$$\mathrm{offdiag}\{a, b, \ldots, y, z\} = \begin{pmatrix} 0 & 0 & \cdots & 0 & a \\ 0 & 0 & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & y & \cdots & 0 & 0 \\ z & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $a, b, \ldots, y, z$ are all complex numbers.

(4) The $(\ell \times \ell)$ matrices $\boldsymbol{e}_{pq}$ (where $1 \leqslant p, q \leqslant \ell$) are defined by

$$(e_{pq})_{jk} = \delta_{jp}\delta_{kq}. \tag{A1}$$

(5) The matrix $j \times j$ matrix $\boldsymbol{K}_j$ is defined by

$$\boldsymbol{K}_j = \mathrm{offdiag}\{1, 1, \ldots, 1\}. \tag{A2}$$

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